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# Computing the Čech cohomology of decomposition spaces

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## Abstract

A line pattern in a free group  $F$  is defined by a malnormal collection of cyclic subgroups. Otal defined a decomposition space  $\mathcal{D}$  associated to a line pattern. We provide an algorithm that computes a presentation for the Čech cohomology of  $\mathcal{D}$ , thought of as a  $F$ -module. This answers a relative version of a question of Epstein about boundaries of hyperbolic groups.

## 1 Introduction

Epstein asked whether there is an algorithm that computes the Čech cohomology of the Gromov boundary of a hyperbolic group  $\Gamma$ , thought of as a  $\Gamma$ -module. Our purpose is to answer a relative version of this question in the case of Otal's decomposition space, a special case of the Bowditch boundary of a relatively hyperbolic group.

Fix a free group  $F$  of rank  $n$  with free basis  $\mathcal{B}$ . Let  $\mathcal{T}$  be the corresponding Cayley graph containing a vertex  $e$  corresponding to the identity in  $F$ . Fix a finite set  $\{w_i\}$  of words in  $F$ . Then the line pattern  $\mathcal{L}$  associated to this set is defined to be the set of lines  $\{gw_i^k\}_k$ ,  $g \in F$ . Let  $\mathcal{D}$  be the associated decomposition space: this is the quotient of  $\partial_\infty F$  by the equivalence relation that identifies the two end points of each line in  $\mathcal{L}$ . These objects are defined in [4]. Let  $q: \partial_\infty F \rightarrow \mathcal{D}$  be the quotient projection. Equivalently,  $\mathcal{D}$  is the Bowditch boundary of the relatively hyperbolic group  $(F, \mathcal{P})$  where  $\mathcal{P}$  is a peripheral family of cyclic subgroups [3]. We present an algorithm that computes the Čech cohomology of  $\mathcal{D}$ .

$F$  acts on  $\partial_\infty F$  and this action descends to an action on  $\mathcal{D}$ . This gives the cohomology groups of  $\mathcal{D}$  the structure of a right  $F$ -module. As a corollary to our main result, we shall see that the Čech cohomology of  $\mathcal{D}$  is finitely presented as an  $F$ -module.

The Čech cohomology of  $\mathcal{D}$  is defined to be the direct limit over open covers  $\mathcal{U}$  that provide successively better combinatorial approximations to  $\mathcal{D}$  of the singular cohomology of the nerve of  $\mathcal{U}$ . In Section 2 we shall see how to associate an open cover of  $\mathcal{D}$  to a finite subtree of  $\mathcal{T}$ . Then refining to a finer open cover of  $\mathcal{D}$  corresponds to taking a larger subtree of  $\mathcal{T}$ . We shall see that the combinatorial properties of this open cover can be read from the Whitehead graphs associated to the subtrees and that any open cover of  $\mathcal{D}$  can be refined to an open cover of this form. These open covers have no triple intersections,

so we immediately see that the Čech cohomology is concentrated in the 0th and 1st dimensions.

In Sections 3 and 4 we show how to compute the 0th and 1st Čech cohomology respectively. Our methods rely on showing that some (large) finite subtree of  $\mathcal{T}$  contains sufficient information to compute the Čech cohomology. This approach is based on the proof of [1, Lemma 4.12]. As corollaries we show that there are algorithms that detect the connectedness of  $\mathcal{D}$  and the triviality of  $\check{H}^1(\mathcal{D}, \mathbb{Z})$ . The former of these corollaries is proved by a different argument in [1].

## 2 Whitehead graphs and open covers

For  $\mathcal{X}$  a subtree of  $\mathcal{T}$ , let  $\text{Wh}(\mathcal{X})$  be the Whitehead graph of  $\mathcal{L}$  at  $\mathcal{X}$  as defined in [1]; briefly, it is a graph with a vertex corresponding to each vertex of  $\mathcal{T}$  adjacent to  $\mathcal{X}$  and an edge connecting a pair of vertices for each line in  $\mathcal{L}$  between that pair. For more information about Whitehead graphs and their applications, see [2].

For  $v \in T$  let  $S^e(v) \subset \partial_\infty \mathcal{T}$  be the shadow of  $v$  from  $e$  as defined in [1]: the set of boundary points  $\xi$  such that the geodesic  $[e, \xi]$  contains  $v$ . These sets are open and closed and the collection of such sets as  $v$  varies in  $\mathcal{T}$  is a basis for the topology on  $\partial_\infty \mathcal{T}$ .

**Lemma 1.** *Let  $\mathcal{X}$  be a finite subtree of  $\mathcal{T}$  containing  $e$ . Then there is a covering of  $\mathcal{D}$  by a collection of open sets  $U_i$  in bijection with the vertices  $a_i$  of  $\text{Wh}(\mathcal{X})$  such that:*

- $q(S^e(a_i)) \subset U_i$ ,
- $U_i \cap U_j \neq \emptyset$  iff there is an edge connecting  $a_i$  and  $a_j$  in  $\text{Wh}(\mathcal{X})$ , and
- there are no triple intersections.

*Proof.* We aim to construct open sets  $V_i$  covering  $\partial_\infty \mathcal{T}$  such that

- $V_i$  contains  $S^e(a_i)$ ,
- $V_i \cap V_j = \emptyset$  iff there is an edge connecting  $a_i$  and  $a_j$  in  $\text{Wh}(\mathcal{X})$ ,
- there are no triple intersections, and
- for each line  $l$  in the line pattern, each  $V_i$  contains either both of  $l^{\pm\infty}$  or neither.

Then the projection of these sets in  $\mathcal{D}$  satisfies the requirements of the lemma.

We build these inductively. For the first step, take  $V_i^0 = S^e(a_i)$ . Then, for each  $i$ , there are finitely many lines in the line pattern passing through  $a_i$ . Add to  $V_i^0$  an open neighbourhood of the end point not in  $V_i^0$  of each such line to obtain  $V_i^1$ . We do this in a way as to ensure that  $V_i^1$  is the union of  $V_i^0$  and finitely many other shadows, that the open sets added are all disjoint, and that no line in the line pattern has an end in two different added sets. This is possible since if a subset of  $\partial_\infty \mathcal{T}$  is a union of finitely many shadows then only finitely many lines in the line pattern have exactly one end in that subset.

After each  $V_i^1$  is defined, continue inductively, ensuring that each  $V_i^k$  is the union of finitely many shadows, so that if a line in the line pattern has one end in  $V_i^k$  then its other end is in  $V_i^{k+1}$ . We can do this without introducing any new intersections, so all intersections correspond to lines from  $S^e(a_i)$  to  $S^e(a_j)$  for some  $i$  and  $j$ , so all intersections correspond to edges in the Whitehead graph and there are no triple intersections.

Then let  $U_i = q(\cup_k V_i^k)$ ; these sets cover  $D$  and have the required properties.  $\square$

For  $\mathcal{X}$  a finite subtree of  $\mathcal{T}$ , we shall denote by  $\mathcal{U}_{\mathcal{X}}$  a cover of  $\mathcal{D}$  associated to  $\mathcal{X}$ . Then if  $\mathcal{X} \subset \mathcal{X}'$ ,  $\mathcal{U}_{\mathcal{X}'}$  can be chosen to be a refinement of  $\mathcal{U}_{\mathcal{X}}$ . Note that refinement between different open covers associated to  $\mathcal{X}$  as in Lemma 1 induces a natural isomorphism between the singular cochain complexes of the nerves of those covers.

It will be convenient to define an open cover associated to the empty subtree of  $\mathcal{X}$ : this is the trivial covering  $\mathcal{U}_{\emptyset} = \{\mathcal{D}\}$ .

**Lemma 2.** *Let  $\mathcal{W}$  be a finite open cover of  $\mathcal{D}$ . Then some refinement of  $\mathcal{W}$  is of the form given in Lemma 1.*

*Proof.* Let  $\mathcal{V}$  be the pullback of  $\mathcal{W}$  to  $\partial_{\infty}\mathcal{T}$ . Consider the set

$$C = \{a \in \mathcal{T} \mid S^e(a) \subset V \text{ for some } V \in \mathcal{V}\}. \quad (1)$$

The collection  $\{S^e(x) \mid x \in \mathcal{T}\}$  is a basis for the topology on  $\partial_{\infty}\mathcal{T}$  so sets of the form  $S^e(a)$ ,  $a \in C$ , cover each  $V \in \mathcal{V}$ . Hence such sets cover  $\partial_{\infty}\mathcal{T}$ .

$\partial_{\infty}\mathcal{T}$  is compact, so there is a finite set of points  $a_1, \dots, a_n$  such that  $\{S^e(a_i)\}$  covers  $\partial_{\infty}\mathcal{T}$  and each  $S^e(a_i)$  is contained in some  $V_{\sigma(i)} \in \mathcal{V}$ . Let  $\mathcal{H}$  be the convex hull of  $\{a_i\} \cup \{e\}$ . Call vertices of  $\mathcal{H}$  adjacent to vertices in  $\mathcal{T} - \mathcal{H}$  boundary vertices. If we take  $\{a_i\}$  to be minimal with its covering property then the set of boundary points of  $\mathcal{H}$  is precisely  $\{a_i\}$ . Let  $\mathcal{X}$  be the subtree of  $\mathcal{H}$  obtained by pruning off its boundary vertices.

Let  $\mathcal{U}_{\mathcal{X}} = \{U_i\}$  be the finite cover of  $\mathcal{D}$  corresponding to  $\mathcal{X}$  as in Lemma 1. Define a new set  $\mathcal{U}'$  of open subsets of  $\mathcal{D}$  by

$$\mathcal{U}' = \{U_i \cap q(V_{\sigma(i)})\}. \quad (2)$$

$\mathcal{U}'$  covers  $\mathcal{D}$  since it covers  $q(S^e(a_i))$  for each  $i$ . It is certainly a refinement of  $\mathcal{W}$  and it is easy to check that it corresponds to  $\mathcal{X}$  in the sense of the statement of Lemma 1.  $\square$

The results of this section together imply the following corollary:

**Corollary 1.**

$$\check{H}^n(\mathcal{D}, \mathbb{Z}) = \varinjlim_{\mathcal{X}} \check{H}^n(\mathcal{U}_{\mathcal{X}}, \mathbb{Z}) \quad (3)$$

with subtrees  $\mathcal{X}$  ordered by inclusion.  $\square$

Hence the Čech cohomology of the decomposition space is determined by the finite Whitehead graphs.

### 3 Computing $\check{H}^0(\mathcal{D}, \mathbb{Z})$

For each element  $[\sigma] \in \check{H}^0(\mathcal{D}, \mathbb{Z})$  there exists a subtree  $\mathcal{X} \subset \mathcal{T}$  such that  $[\sigma]$  is represented by some

$$\sigma \in \check{H}^0(\mathcal{U}_{\mathcal{X}}, \mathbb{Z}) = \ker(d^0: \check{C}^0(\mathcal{U}_{\mathcal{X}}, \mathbb{Z}) \rightarrow \check{C}^1(\mathcal{U}_{\mathcal{X}}, \mathbb{Z})); \quad (4)$$

such an element is an assignment of an integer to each connected component of  $\text{Wh}(\mathcal{X})$ . In this situation we shall say that  $[\sigma]$  is supported on  $\mathcal{X}$  and we shall refer to the minimal such subtree as the support of  $[\sigma]$ . A unique minimal such subtree exists by the following lemma:

**Lemma 3.** *Suppose that  $[\sigma] \in \check{H}^0(\mathcal{D}, \mathbb{Z})$  is supported on  $\mathcal{X}_1$  and on  $\mathcal{X}_2$ . Then it is also supported on  $\mathcal{X}_1 \cap \mathcal{X}_2$ .*

*Proof.* We prove this by induction on the number of vertices in the symmetric difference of  $\mathcal{X}_1$  and  $\mathcal{X}_2$ . If the symmetric difference is non-empty then without loss of generality  $\mathcal{X}_1$  has a leaf  $v$  that is not contained in  $\mathcal{X}_2$ . It is easy to see that  $[\sigma]$  is supported on  $\mathcal{X}_1 - v$ .  $\square$

As discussed in the introduction,  $F$  acts on  $\mathcal{D}$  by homeomorphisms, giving the Čech cohomology the structure of a right  $F$ -module. In terms of Whitehead graphs, any  $g \in F$  induces a map

$$g: \check{H}^0(\mathcal{U}_{\mathcal{X}}, \mathbb{Z}) \rightarrow \check{H}^0(g^{-1}\mathcal{U}_{\mathcal{X}}, \mathbb{Z}) = \check{H}^0(\mathcal{U}_{g^{-1}\mathcal{X}}, \mathbb{Z}); \quad (5)$$

this map takes an element represented by a  $\mathbb{Z}$ -labelling of the connected components of  $\text{Wh}(\mathcal{X})$  to the element represented by the translate by  $g^{-1}$  of this diagram.

We now aim to find an algorithm that computes a presentation for this  $F$ -module. First we describe an algorithm that computes a generating set. The argument is loosely based on the proof of lemma 4.12 in [1].

**Theorem 1.** *There exists a finite number  $N$ , computable from  $n$  and  $\mathcal{L}$ , such that  $\check{H}^0(\mathcal{D}, \mathbb{Z})$  is generated as an abelian group by 0-cycles supported on subtrees of  $\mathcal{T}$  with at most  $N$  vertices.*

*Proof.* Let  $[\sigma]$  be a 0-cycle supported on a subtree  $\mathcal{X}$  of  $\mathcal{T}$  with more than  $N$  vertices, where  $N$  is to be chosen later. Then by induction it is sufficient to show that  $[\sigma]$  can be written as the sum of 0-cycles supported on strictly smaller subtrees. The idea is that if  $N$  is large enough then there will be two vertices in  $\mathcal{X}$  at which  $[\sigma]$  looks similar and cutting out everything between these vertices allows us to split  $[\sigma]$  into strictly smaller summands.

$\check{H}^0(\mathcal{U}_{\mathcal{X}}, \mathbb{Z})$  is generated by 0-cycles represented by Whitehead diagrams at  $\mathcal{X}$  with one connected component labelled 1 and the others labelled 0; we can assume without loss of generality that  $[\sigma]$  is such a 0-cycle. Then  $[\sigma]$  can be thought of as a partition of  $\text{Wh}(\mathcal{X})$  into a connected component and its complement. An example of such a partition is shown pictorially in figure 1.

Suppose that  $N$  is large enough that any subtree of  $\mathcal{T}$  with more than  $N$  vertices is guaranteed to contain an embedded arc of length at least  $M + 2$ , where  $M$  is a computable function of  $n$  and  $\mathcal{L}$  to be chosen later. Then let  $v_1, \dots, v_M$  be the interior vertices of such an embedded arc in  $\mathcal{X}$ . Traversing

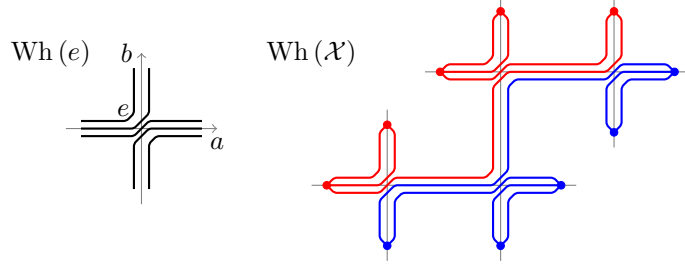


Figure 1: A partition of a disconnected Whitehead graph into two connected components. An assignment of the integer 1 to the red part and 0 to the blue part represents an element of  $\check{H}^0(\mathcal{D}, \mathbb{Z})$ . In this example  $F$  is free on two generators and the line pattern is generated by the word  $a^2bab$ .

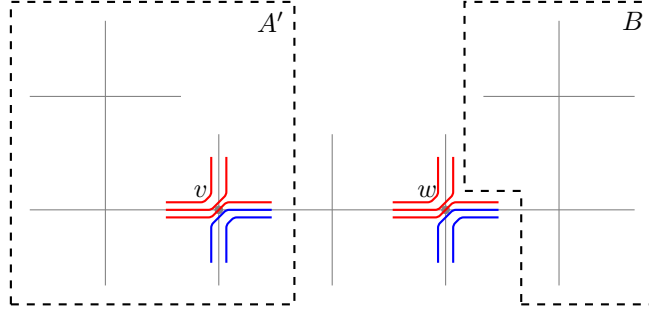


Figure 2:  $v$  and  $w$  are chosen so that  $\sigma$  induces the same partition on  $\text{Wh}(v)$  as on  $\text{Wh}(w)$ . Then  $\tau$  is an element of  $\check{H}^0(\mathcal{U}_{A \cup vw^{-1}B}, \mathbb{Z})$  chosen to induce the same partition on  $\text{Wh}(A')$  as  $\sigma$  does.

this arc in the direction from  $v_1$  to  $v_M$ , record for each vertex  $v_i$  an ordered pair  $(s_i, t_i)$  of elements of  $\mathcal{B}^\pm$ , where  $s_i$  labels the incoming edge at  $v_i$  of the embedded arc at and  $t_i$  labels the outgoing edge.

Suppose that  $M$  is large enough that at least  $K$  of these pairs are equal. Here  $K$  is a computable function of  $\mathcal{L}$  to be chosen later. Then let  $v_{i_1}, \dots, v_{i_K}$  be vertices with equal associated edge pairs. The edges of each  $\text{Wh}(v_{i_j})$  extend to edges in  $\text{Wh}(\mathcal{L})$ , hence the partition of  $\text{Wh}(\mathcal{L})$  associated to  $[\sigma]$  gives a partition of the edges of  $\text{Wh}(v_{i_j})$  into a subset and its complement.

Treating the  $v_{i_j}$  as elements of  $F$ , the translation of  $\text{Wh}(v_{i_j})$  by  $v_{i_j}^{-1}$  gives a partition on the edges of  $\text{Wh}(e)$ . There is a finite number of such partitions; let  $K$  be greater than that number. Then we obtain  $v, w = g(v) \in \{v_{i_1}, \dots, v_{i_K}\}$  such that these translates of the associated partitions agree.

Now we define two disjoint subsets of  $\mathcal{X}$ . Let  $A$  be the vertices  $u \neq v$  of  $\mathcal{X}$  such that the geodesic in  $\mathcal{T}$  from  $w$  to  $u$  passes through  $v$ , and let  $B$  be the same with the rôles of  $v$  and  $w$  reversed. Without loss of generality,  $A$  contains at least as many vertices as  $B$  does. Then let  $A' = A \cup \{v\}$ . See figure 2.

We now cancel off the part of  $[\sigma]$  supported on  $A$ . Let  $\mathcal{Y} = A' \cup vw^{-1}B$ , where we treat  $v$  and  $w$  as elements of the group  $F$ . As described above,  $\sigma$  induces a partition on the edges of  $\text{Wh}(u)$  for each vertex  $u$  in  $A'$  and  $B$ , and hence by translation on  $\text{Wh}(u)$  for each vertex  $u$  in  $\mathcal{Y}$ . The partitions at

$v \in A'$  and the vertex of  $vw^{-1}B$  adjacent to  $v$  are consistent with respect to the splicing map, so we obtain a partition of the graph  $\text{Wh}(\mathcal{Y})$ . Hence we can define  $\tau \in \check{H}^0(\mathcal{Y}, \mathbb{Z})$  to be a cycle represented by assigning the integer 1 to one component of  $\text{Wh}(\mathcal{Y})$  and 0 to the others in a way that agrees at the vertices of  $A'$  with the labelling of components represented by  $\sigma$ .

Then  $[\tau]$  is supported on  $\mathcal{Y}$ , which contains at most  $|A'| + |B| < |\mathcal{X}|$  vertices, and  $[\sigma] - [\tau]$  is supported on  $(\mathcal{X} - A) \cup g^{-1}B$ , which has fewer vertices than  $\mathcal{X}$  since  $A'$  has more vertices than  $B$ .  $\square$

**Remark 1.** We can give bounds on the function  $N(\mathcal{L}, n)$  explicitly: let  $k$  be the number of edges in  $\text{Wh}(e)$ ; this is equal to the sum of the lengths of the words that generate  $\mathcal{L}$ . Then:

$$N = (2n)^{(2n)^2(2^k+1)+1} (1 + 2n) + 1 \quad (6)$$

**Corollary 2.**  $\check{H}^0(\mathcal{D}, \mathbb{Z})$  has a computable finite generating set.

*Proof.* Theorem 1 implies that  $\check{H}^0(\mathcal{D}, \mathbb{Z})$  is generated as an  $F$ -module by the set of 0-cycles supported on subtrees of the ball of radius  $N$  centred at  $e$ . If  $\mathcal{X}$  is this ball then  $\check{H}^0(\mathcal{U}_{\mathcal{X}}, \mathbb{Z})$  has a computable finite generating set as a  $\mathbb{Z}$ -module, since  $\text{Wh}(\mathcal{X})$  can be partitioned into its connected components algorithmically.  $\square$

**Corollary 3.** There is an algorithm that determines whether or not  $\mathcal{D}$  is connected.

This corollary is proved by a different argument in [1]. In that paper it is shown that, after simplifying  $\text{Wh}(e)$  as much as possible using Whitehead moves,  $\mathcal{D}$  is connected if and only if  $\text{Wh}(e)$  is connected.

*Proof.*  $\mathcal{D}$  is connected if and only if  $\check{H}^0(\mathcal{D}, \mathbb{Z})$  is generated by the cochain supported on the trivial covering that assigns the integer 1 to the only open set in that covering; in this case it is isomorphic to  $\mathbb{Z}$  with trivial  $F$  action. Equivalently,  $\mathcal{D}$  is connected if and only if any  $\sigma \in \check{H}^0(\mathcal{U}_{\mathcal{X}}, \mathbb{Z})$  is represented by the assignment of the same integer to each component of  $\text{Wh}(\mathcal{X})$  for all subtrees  $\mathcal{X} \subset \mathcal{T}$ . It is sufficient to check this on a generating set, and we have already shown that  $\check{H}^0(\mathcal{D}, \mathbb{Z})$  has a computable finite generating set.  $\square$

We now have an algorithm that gives a finite set  $[\sigma_1], \dots, [\sigma_k]$  of cohomology classes that generate  $\check{H}^0(\mathcal{D}, \mathbb{Z})$  as a right  $F$ -module. This is equivalent to a surjection  $(p: \mathbb{Z}F^k \rightarrow \check{H}^0(\mathcal{D}, \mathbb{Z}))$  of right  $F$ -modules. Let  $e_i$  be the  $i$ th basis vector in the free module, and let it be mapped to  $[\sigma_i]$  under  $p$ . To complete the computation of a presentation for  $\check{H}^0(\mathcal{D}, \mathbb{Z})$  we need an algorithm that computes a generating set for the kernel of  $p$ .

For each  $[\sigma_i]$  let  $\mathcal{X}_i$  be the support of  $[\sigma_i]$  is supported. A general element  $x \in \mathbb{Z}F^k$  is of the form

$$x = \sum_{i,j} n_{ij} (e_j g_{ij}), \text{ where } n_{ij} \in \mathbb{Z}, g_{ij} \in F. \quad (7)$$

Define the support of  $x$  to be

$$\text{hull} \left( \bigcup_{i,j} g_{ij}^{-1} \mathcal{X}_j \right). \quad (8)$$

Note that the support of  $px$  is contained in the support of  $x$ .

We can now state and prove a theorem that shows that the kernel of  $p$  is generated by elements of bounded size, in the same way that Theorem 1 shows that  $\check{H}^0(\mathcal{D}, \mathbb{Z})$  is generated by elements of bounded size.

**Theorem 2.**  *$\ker p$  is generated as an abelian group by elements whose supports have at most  $N$  vertices, where  $N$  is a computable function of  $\mathcal{L}$  and  $n$ .*

*Proof.* Our approach here is similar to that in the proof of Theorem 1: we show that—for sufficiently large (computable)  $N$ —an element of  $\ker p$  supported on a set with more than  $N$  vertices can be written as the sum of two elements of  $\ker p$  supported on strictly smaller sets.

Let  $D$  be the maximum of the diameters of the  $\mathcal{X}_i$ . Let a ball of diameter  $D$  contain  $L$  vertices.

From the proof of Theorem 1 it is clear that in picking a preimage  $x$  under  $p$  of an element  $[\sigma] \in \check{H}^0(\mathcal{D}, \mathbb{Z})$  it might well be necessary for the support of  $x$  to be strictly larger than the support of  $[\sigma]$ . We will need to be able to bound the size of the support of  $x$  for  $[\sigma]$  supported on a ball of radius at most  $D$ . We deal with this first.

With some care, the proof of Theorem 1 gives an explicit bound. At each step, the cochain is split into two pieces, each supported on a set with strictly fewer vertices. Hence, since  $[\sigma]$  is supported on a set with  $L$  vertices, it can certainly be written as a  $\mathbb{Z}$ -linear combination of at most  $2^L$  elements of our generating set. So if each generator has at most  $M$  vertices, any  $[\sigma]$  supported on a ball of radius  $D$  has a preimage supported on a set with at most  $2^L M$  vertices. By construction, this set can be taken to be connected. Let  $K = 2^L M$ .

Let  $N$  be large enough that any subtree  $\mathcal{X}$  of  $\mathcal{T}$  with at least  $N$  vertices contains a vertex  $v$  such that  $\mathcal{X} - v$  is the union of two (disconnected) subgraphs of  $\mathcal{X}$  each with at least  $K + L$  vertices. For example, this holds if  $\mathcal{X}$  is guaranteed to contain an embedded arc of length at least  $2(K + L) + 1$ . Then suppose that some relator  $x \in \ker p$  is supported on a subtree  $\mathcal{X} \subset \mathcal{T}$  with at least  $N$  vertices. Let  $v$  be as in the definition of  $N$ . Then we aim to divide  $x$  as the sum of two smaller relators by cutting at  $v$ .

$x$  is of the form of equation 7 and is such that  $g_{ij}^{-1} \mathcal{X}_j \subset \mathcal{X}$  for each pair  $i, j$ . Let  $A$  and  $B$  be the two components of  $\mathcal{X} - v$  as described above, and let  $C$  be the ball in  $\mathcal{X}$  of radius  $D$  centred at  $v$ . Let  $y \in \mathbb{Z}F^k$  be the sum of those summands of  $x$  in equation 7 whose supports are contained in  $A$ . Then the support of  $y$  is a subset of  $A$  and the support of  $x - y$  is a subset of  $B \cup C$ .

Roughly,  $y$  and  $x - y$  will be the two desired smaller relators whose sum is  $x$ . However  $py \neq 0$ , so we shall need to add a small correction term. In order to ensure that the correction term is indeed small (in the sense of having small support) we use Lemma 3.

Since  $py = -p(x - y)$ ,  $py$  is supported on  $A \cap (B \cup C) = A \cap C$ . This is a subtree of a tree of diameter  $2D$ , so by assumption  $py$  has a preimage  $w$  under  $p$  that is supported on a set with at most  $K$  vertices. Then  $p(y - w) = 0$  and  $x = (y - w) + (x - y + w)$  so it remains to show that  $y - w$  and  $x - y + w$  have strictly smaller supports than  $x$ . But the support of  $x$  has  $|A| + |B| + 1$  vertices, while  $y - w$  and  $x - y + w$  are supported on sets with at most  $|A| + K$  and  $|B| + |C| + K$  vertices respectively.  $|A|$  and  $|B|$  have at least  $K + |C|$  vertices, so this completes the proof.  $\square$



**Corollary 4.** *There is an algorithm that computes a finite presentation for  $\check{H}^0(\mathcal{D}, \mathbb{Z})$ .*

*Proof.* Note that  $F$  acts on  $\mathbb{Z}F^k$  by translation in the sense that if the support of  $x \in \mathbb{Z}F^k$  is  $\mathcal{X}$  then the support of  $xg$  is  $g^{-1}\mathcal{X}$ . Hence if  $N$  is as in the statement of Theorem 2 then that theorem shows that  $\ker p$  is generated as an  $F$ -module by those of its elements that are supported on a ball of radius  $N$  ball centred at  $e$ .

In other words,  $\ker p$  is generated by its intersection with the set of those  $\mathbb{Z}$ -linear combinations of translates of the  $\{e_i\}$  by  $F$  whose supports are contained in this ball of radius  $N$ . To find all such linear combinations is simply to solve a finite dimensional  $\mathbb{Z}$ -linear equation, which can be done algorithmically, for example using Smith normal form.  $\square$

## 4 Computing $\check{H}^1(\mathcal{D}, \mathbb{Z})$

$\check{H}^1(\mathcal{U}_{\mathcal{X}}, \mathbb{Z})$  is the quotient of  $\check{C}^1(\mathcal{U}_{\mathcal{X}}, \mathbb{Z})$  by  $d\check{C}^0(\mathcal{U}_{\mathcal{X}}, \mathbb{Z})$ , since  $\check{C}^2(\mathcal{U}_{\mathcal{X}}, \mathbb{Z})$  is trivial. Since taking direct limits of families of  $\mathbb{Z}$ -modules is an exact functor,  $\check{H}^1(\mathcal{D}, \mathbb{Z})$  is also a quotient:

$$0 \longrightarrow d\varinjlim_{\mathcal{X}} \check{C}^0(\mathcal{U}_{\mathcal{X}}, \mathbb{Z}) \longrightarrow \varinjlim_{\mathcal{X}} \check{C}^1(\mathcal{U}_{\mathcal{X}}, \mathbb{Z}) \longrightarrow \check{H}^1(\mathcal{D}, \mathbb{Z}) \longrightarrow 0$$

is exact. As in the previous section, each of these abelian groups can be endowed with the structure of an  $F$ -module so that the homomorphisms in the short exact sequence are homomorphisms of  $F$ -modules.

Now finding a presentation for  $\check{H}^1(\mathcal{D}, \mathbb{Z})$  is equivalent to finding a presentation for  $\varinjlim_{\mathcal{X}} \check{C}^1(\mathcal{U}_{\mathcal{X}}, \mathbb{Z})$  and a generating set for  $d\varinjlim_{\mathcal{X}} \check{C}^0(\mathcal{U}_{\mathcal{X}}, \mathbb{Z})$ . We present an algorithm that does the former in theorems 3 and 4 and an algorithm that does the latter in Lemma 5.

As in the previous section, cochains have a convenient representation in terms of the Whitehead graph. A 1-cochain (with respect to an open cover  $\mathcal{U}$ ) is a map that associates an integer to each pair  $U_1, U_2 \in \mathcal{U}$  with  $U_1 \cap U_2 \neq \emptyset$ . Equivalently, if  $\mathcal{U}$  is the open cover associated to a Whitehead graph  $\text{Wh}(\mathcal{X})$ , this is the assignment of an integer to each edge in the Whitehead graph, with the restriction that if two edges connect the same pair of vertices then they are assigned the same integer. Refinement to the open cover associated to a larger Whitehead graph preserves the labelling of the old edges, and assigns the integer 0 to each new edge.

**Theorem 3.** *There is a computable function  $N$  of  $\mathcal{L}$  and  $n$  so that  $\varinjlim \check{C}^1(\mathcal{U}_{\mathcal{X}}, \mathbb{Z})$  is generated as an abelian group by elements supported on sets with fewer than  $N$  vertices.*

*Proof.* A subset  $\mathcal{X} \subset \mathcal{C}$  gives a partition  $P_{\mathcal{X}}$  on the edges of  $\text{Wh}(v)$  for each vertex  $v$  of  $\mathcal{X}$ : in this partition, two edges are related if those edges extend to edges between the same pair of vertices in  $\text{Wh}(\mathcal{X})$ . For each element  $a \in \mathcal{B}^{\pm}$  and partition  $P$  on the edges of  $\text{Wh}(e)$  there exists a subset  $\mathcal{X} \subset \text{hull}(e \cup S^e(a)) \cap \mathcal{T}$  such that  $P$  is at least as fine as  $P_{\mathcal{X}}$ . Let  $\mathcal{X}_{(a,P)}$  be a minimal such subset; it is easy to see that it is contained in any other subset with this property.

Let  $N$  be the maximum number of vertices in any  $\mathcal{X}_{(a,P)}$ . We now prove that  $\varinjlim \check{C}^1(\mathcal{U}_{\mathcal{X}}, \mathbb{Z})$  is generated as an abelian group by elements supported on sets with at most  $N$  vertices.

Let  $[\sigma]$  be a 1-cochain supported on  $\mathcal{X}$  and let  $\sigma \in \check{C}^1(\mathcal{U}_{\mathcal{X}}, \mathbb{Z})$  represent  $[\sigma]$ . Let  $v$  be a leaf of  $\mathcal{X}$ . Then  $\sigma$  defines a partition  $P$  on the edges of  $\text{Wh}(v)$  by relating two edges if they are assigned the same integer by  $\sigma$ .  $P_{\mathcal{X}}$  is at least as fine as  $P$ . By translating  $P$  by  $v^{-1}$  (considering the vertex as an element of the group  $F$ ) we obtain a partition on  $\text{Wh}(e)$ , which we shall also denote by  $P$ . Let  $a \in \mathcal{B}$  be the label on the edge connecting  $v$  to the rest of  $\mathcal{X}$ .

By the definition of  $N$ ,  $\mathcal{X}_{(a,P)}$  has at most  $N$  vertices. Let  $\tau$  be a 1-cochain supported on this set that assigns to each edge of  $\text{Wh}(e)$  the same integer that  $\sigma v$  does; note that  $\tau$  satisfies the requirement that if two edges connect the same pair of vertices in the Whitehead graph then they are assigned the same integer. Since  $v\mathcal{X}_{(a,P)} \subset \mathcal{X}$ ,  $\tau v^{-1}$  is supported on  $\mathcal{X}$  and then it is easy to see that  $\sigma - \tau v^{-1}$  is supported on  $\mathcal{X} - v$ . Proceeding by induction on the number of vertices in the support of  $\sigma$  we obtain the required result.  $\square$

This immediately implies the following corollary:

**Corollary 5.**  $\varinjlim \check{C}^1(\mathcal{U}_{\mathcal{X}}, \mathbb{Z})$  is generated as an  $F$ -module by those of its elements that are supported on a ball centred at  $e$  of computable finite diameter.  $\square$

To proceed to compute a set of relators for  $\varinjlim \check{C}^1(\mathcal{U}_{\mathcal{X}}, \mathbb{Z})$  we require the following lemma, which is analagous to Lemma 3.

**Lemma 4.** Let  $D$  be the maximum of the diameters of the supports of the generators computed in Theorem 3. Suppose that  $[\sigma] \in \varinjlim \check{C}^1(\mathcal{U}_{\mathcal{X}}, \mathbb{Z})$  is supported on  $\mathcal{X}_1$  and on  $\mathcal{X}_2$  where  $\mathcal{X}_1$  and  $\mathcal{X}_2$  are subtrees of  $\mathcal{C}$  with non-trivial intersection. Then  $[\sigma]$  it is also supported on a  $D$ -neighbourhood of  $\mathcal{X}_1 \cap \mathcal{X}_2$ .

*Proof.*  $[\sigma]$  is represented by a labelling of the edges of  $\text{Wh}(\mathcal{X}_1 \cup \mathcal{X}_2)$  by integers such that each edge that does not pass through  $\mathcal{X}_1 \cap \mathcal{X}_2$  is labelled by 0. Any such 1-cycle is supported on a  $D$ -neighbourhood of this subset.  $\square$

**Theorem 4.** There is an algorithm that computes a set of relators for the  $F$ -module  $\varinjlim \check{C}^1(\mathcal{U}_{\mathcal{X}}, \mathbb{Z})$  with respect to the basis computed by the algorithm of Theorem 3.

*Proof.* The proof of Theorem 2 works here too; Lemma 4 has a weaker hypothesis than Lemma 3 does but this makes no difference to the proof.  $\square$

Theorems 3 and 4 together give an algorithm that computes a finite presentation for  $\varinjlim \check{C}^1(\mathcal{U}_{\mathcal{X}}, \mathbb{Z})$ .  $\check{H}^1(\mathcal{D}, \mathbb{Z})$  is the quotient of this abelian group by the image under the boundary map of  $\varinjlim \check{C}^0(\mathcal{U}_{\mathcal{X}}, \mathbb{Z})$ , so it remains to show that this image has a computable generating set.

**Lemma 5.**  $d \varinjlim \check{C}^0(\mathcal{U}_{\mathcal{X}}, \mathbb{Z})$  is generated as an  $F$ -module by those of its elements that are supported on  $\text{Wh}(e)$ .

*Proof.*  $\check{C}^0(\mathcal{U}_{\mathcal{X}}, \mathbb{Z})$  is generated as an abelian group by those elements that are supported on  $\text{Wh}(v)$  for some  $v \in \mathcal{X}$ , and if  $[\sigma]$  is supported on  $\mathcal{X}$  then so is  $d[\sigma]$ .  $\square$

Putting the results of this section together we conclude:

**Theorem 5.** *There is an algorithm that determines a presentation for the  $F$ -module  $\check{H}^1(\mathcal{D}, \mathbb{Z})$ .*  $\square$

**Corollary 6.** *There is an algorithm that determines whether or not  $\check{H}^1(\mathcal{D}, \mathbb{Z})$  is trivial.*

*Proof.* Since  $\check{H}^1(\mathcal{D}, \mathbb{Z})$  has a computable finite generating set, it is sufficient to be able to determine whether or not each generator is trivial. But each  $\check{H}^1(\mathcal{U}_X, \mathbb{Z})$  includes injectively into  $\check{H}^1(\mathcal{D}, \mathbb{Z})$ , so it is sufficient to be able to determine whether or not a given element of some  $\check{H}^1(\mathcal{U}_X, \mathbb{Z})$  is trivial; that is, whether or not it is in the image of  $d$ . But the problem of determining whether or not such an element has a preimage under  $d$  is equivalent to determining whether or not some finite dimensional  $\mathbb{Z}$ -linear equation has a solution, so can be done algorithmically.  $\square$

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